Conformal Field Theory and Gravity

Solutions to Problem Set 4

Fall 2024

1. Large-N gauge theory

(a) We can find expectations through

$$\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle = \frac{1}{Z}\frac{(-1)^n}{N^n}\frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)}\bigg|_{J=0} \tag{1}$$

The diagrammatic expansion of $Z[J]$ now contains new vertices. However, as we saw in the previous exercise, the number edges meeting at each interactions vertex does not affect the power of N in the scaling of the diagram. Let us now consider the generator of connected diagrams (this will allow us to only reason in terms of single bubble diagrams, and not disconnected contributions)

$$W[J] = \log Z[J] = S_0[J] + S_1[J] + ... \tag{2}$$

where S_n denotes the sum of connected bubble diagrams with genus n. Hence, if we use the scaling determined in the previous question:

$$W[J] = N^2 F_0[J,\lambda] + N^0 F_1[J,\lambda] + \mathcal{O}(N^{-2}) \tag{3}$$

where the dominant contribution corresponds to planar diagrams.

(b) The connected two point function is given by

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{conn} = \frac{1}{N^2}\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = \frac{\delta^2 F_0[J,\lambda]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} + \mathcal{O}(N^{-2}) \tag{4}$$

The disconnected part of the two-point function is

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{disc} = \langle \mathcal{O}(x_1)\rangle\langle \mathcal{O}(x_2)\rangle = N^2\frac{\delta F_0[J,\lambda]}{\delta J(x_1)}\frac{\delta F_0[J,\lambda]}{\delta J(x_2)}\bigg|_{J=0} + \mathcal{O}(N^0) \tag{5}$$

Hence

$$\frac{\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{conn}}{\langle \mathcal{O}(x_1)\rangle\langle \mathcal{O}(x_2)\rangle} \sim N^{-2} \tag{6}$$

(c) Using 1, the functional derivatives bring down the highest power of N when they all act on the factor of $\exp\{N^2 F_0[J_i]\}$.
Therefore, we get

$$\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle = \frac{(-1)^n}{N^n}\prod_j \frac{\delta(N^2 F_0[J_i])}{\delta J_j}\bigg|_{J_i=0}(1+\mathcal{O}(N^{-2})) = \prod_i \langle \mathcal{O}_i\rangle(1+\mathcal{O}(N^{-2})) \tag{7}$$

If n does not stay finite but grows with N, then the number of non-planar diagrams also grows with N, therefore this counting breaks since from simple combinatoric arguments there are many more non-planar than planar diagrams for large N.

(d) Now we have

$$\langle \tilde{\mathcal{O}}_i \rangle \sim \left. \frac{\delta (F_0[\tilde{J}_i])}{\delta \tilde{J}_j} \right|_{\tilde{J}_i=0} = 0 \tag{8}$$

Since F_0 is the only term in the exponent of Z multiplied by a positive power of N, the highest power of N is obtained by bringing down the largest number of F_0 factors. However, now at least two derivatives need to act on F_0 in order to get a non-vanishing expression. In fact, the largest contribution arises when there are exactly two derivatives acting on each F_0. Using the fact that $\langle \tilde{\mathcal{O}}_i \tilde{\mathcal{O}}_j \rangle = \frac{\delta^2 F_0[\tilde{J}]}{\delta \tilde{J}_i \delta \tilde{J}_j}$, we get

$$\langle \prod_i \tilde{\mathcal{O}}_i \rangle = \frac{1}{N^n} \prod_{i<j}^{\text{Wick}} N^2 \left. \frac{\delta^2 F_0[\tilde{J}]}{\delta \tilde{J}_i \delta \tilde{J}_j} \right|_{\tilde{J}=0} (1 + \mathcal{O}(N^{-2})) = \prod_{i<j}^{\text{Wick}} \langle \tilde{\mathcal{O}}_i \tilde{\mathcal{O}}_j \rangle (1 + \mathcal{O}(N^{-2})) \tag{9}$$

where the $\prod_{i<j}^{\text{Wick}}$ ensures all possible indices are considered exactly once. Note that if n is odd then correlations go to 0 for large N.

2. Polyakov action and Virasoro modes

(a) Begin with the action

$$S[X,e] = \frac{1}{2} \int d\tau (e^{-1} \dot{X}^2 - e m^2) \tag{10}$$

By the Euler-Lagrange equations for e

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{e}} \right) - \frac{\partial \mathcal{L}}{\partial e} = e^{-2} \dot{X}^2 + m^2 = 0 \tag{11}$$

The conjugate momentum is given by

$$p_\mu = \frac{\partial}{\partial \dot{X}^\mu} = e^{-1} \dot{X}_\mu \tag{12}$$

Thus, the equation of motion is simply the mass-shell condition $p^2 + m^2 = 0$. For timelike vector $\dot{X}^2 < 0$, we solve for e

$$e = \frac{1}{m} \sqrt{-\dot{X}^2} \tag{13}$$

Inserting it back in the action gives

$$S[X] = -m \int d\tau \sqrt{-\dot{X}^2} \tag{14}$$

(b) Varying with respect to $g^{\mu\nu}$ as one usually does in GR, we obtain that

$$\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} = \sqrt{-g} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2} \sqrt{-g} g_{\alpha\beta} (g^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} - (p-1)) = 0 \tag{15}$$

This can be written as

$$\gamma_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}(g^{\gamma\delta}\gamma_{\gamma\delta} - (p-1)) = 0 \tag{16}$$

Contracting with $g^{\alpha\beta}$ and noting that $g^{\alpha\beta}g_{\alpha\beta} = p+1$, we obtain

$$\frac{1-p}{2} g^{\alpha\beta}\gamma_{\alpha\beta} = -\frac{(p+1)(p-1)}{2} \tag{17}$$

or equivalently

$$g^{\alpha\beta}\gamma_{\alpha\beta} = p+1 \tag{18}$$

Plugging this into (16), we have

$$g_{\alpha\beta} = \gamma_{\alpha\beta} \tag{19}$$

so that

$$S = -\frac{T}{2}\int d^{p+1}\sigma\sqrt{-\det\gamma}((p+1)-(p-1)) = -T\int d^{p+1}\sigma\sqrt{-\det\gamma} \tag{20}$$

(c) From the mode expansion we obtain that

$$\partial_- X^\mu = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-in\sigma^-} \tag{21}$$

where we used $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$. Then we can write $T_{--}(\sigma^-)$ as

$$T_{--}(\sigma^-) = -\frac{1}{2}\sum_{m,n\in\mathbb{Z}} (\alpha_m \cdot \alpha_n) e^{-i(m+n)\sigma^-} \tag{22}$$

The story for T_{++} is similar by exchanging $\sigma^- \to \sigma^+$ and $\alpha_m^\mu \to \tilde{\alpha}_m^\mu$. The ℓ_n are computing using

$$\begin{aligned}\ell_n &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{--}(\sigma)e^{-in\sigma} = \frac{1}{4\pi}\sum_{m,p\in\mathbb{Z}}(\alpha_m\cdot\alpha_p)\int d\sigma e^{i(m+p-n)\sigma} \\ &= \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_m\cdot\alpha_{n-m}\end{aligned} \tag{23}$$

Similarily,

$$\tilde{\ell}_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}(\tilde{\alpha}_m\cdot\tilde{\alpha}_{n-m}) \tag{24}$$

(d) Using the Poisson bracket relation,

$$\begin{aligned}\{\ell_m, \alpha_n^\mu\} &= \frac{1}{2}\sum_{p\in\mathbb{Z}}\{\alpha_p\cdot\alpha_{m-p}, \alpha_n^\mu\} = \frac{1}{2}\sum_{p\in\mathbb{Z}}\eta_{\nu\rho}(\alpha_p^\nu\{\alpha_{m-p}^\rho, \alpha_n^\mu\} + \{\alpha_p^\nu, \alpha_n^\mu\}\alpha_{m-p}^\rho) \\ &= -\frac{i}{2}\sum_{p\in\mathbb{Z}}\eta_{\nu\rho}((m-p)\eta^{\rho\mu}\alpha_p^\nu\delta_{m-p+n,0} + p\eta^{\nu\mu}\alpha_{m-p}^\rho\delta_{p+n,0}) \\ &= -\frac{i}{2}(-n\alpha_{m+n}^\mu - n\alpha_{m+n}^\mu) = in\alpha_{m+n}^\mu\end{aligned} \tag{25}$$

Thus

$$
\begin{aligned}
\{\ell_m, \ell_n\} &= \frac{1}{2} \sum_{p \in \mathbb{Z}} \eta_{\mu\nu} (\alpha^{\mu}_{n-p} \{\ell_m, \alpha^{\nu}_{p}\} + \{\ell_m, \alpha^{\mu}_{n-p}\} \alpha^{\nu}_{p}) \\
&= \frac{i}{2} \sum_{p \in \mathbb{Z}} \eta_{\mu\nu} (p \alpha^{\mu}_{n-p} \alpha^{\nu}_{m+p} + (n-p) \alpha^{\mu}_{m+n-p} \alpha^{\nu}_{p}) \\
&= \frac{i}{2} \sum_{p \in \mathbb{Z}} \eta_{\mu\nu} ((p-m) \alpha^{\mu}_{m+n-p} \alpha^{\nu}_{p} + (n-p) \alpha^{\mu}_{m+n-p} \alpha^{\nu}_{p}) \\
&= \frac{i}{2} (n-m) \sum_{p \in \mathbb{Z}} (\alpha_{m+n-p} \cdot \alpha_{p}) = -i(m-n) \ell_{m+n}
\end{aligned}
\tag{26}
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